# An Inequality for Subadditive Functions on a Distributive Lattice, with Application to Determinantal Inequalities*

KY FAN
*University of California*
*Santa Barbara, California*

1. A real-valued function $f$ defined on a lattice $L$ will be called *subadditive,* if

$$f(x) + f(y) \geqslant f(x \vee y) + f(x \wedge y) \tag{1}$$

holds for any two elements $x, y$ of $L$.

The purpose of this note is to present a new inequality for subadditive functions on a distributive lattice, with application to inequalities concerning principal minors of various matrices.

2. For a square matrix $A$ of order $n$ and for a subset $\alpha$ of the set $\{1, 2, \ldots, n\}$, we shall denote by $A(\alpha)$ the principal minor of $A$ formed by the rows and columns with indices contained in $\alpha$. For the empty set $\varnothing$, we define $A(\varnothing) = 1$.

Let $A$ be a square matrix of order $n$. If $A$ is either a positive definite Hermitian matrix, or an $M$ matrix [11, 12] (i.e., $A$ is of the form $A = \rho I - B$, where $B$ is a real matrix with nonnegative elements, $I$ is the identity matrix, and $\rho$ is a positive number greater than the absolute value of every eigenvalue of $B$), or a totally positive matrix [8, p. 85] (i.e., a real matrix whose minors, principal or not, are all positive), it is known [8, p. 111; 9] that the inequality

$$A(\alpha)A(\beta) \geqslant A(\alpha \cap \beta)A(\alpha \cup \beta) \tag{2}$$

holds for any two subsets $\alpha, \beta$ of the set $\{1, 2, \ldots, n\}$. In other words, the function $f$ defined by

* Work supported in part by the National Science Foundation, Grant GP-5578.

*Linear Algebra and Its Applications* 1, 33–38 (1968)

$$f(\alpha) = \log A(\alpha), \qquad \text{for} \quad \alpha \subset \{1, 2, \ldots, n\}, \tag{3}$$

is a subadditive function on the lattice of subsets of $\{1, 2, \ldots, n\}$.

Let $A = (a_{ij})$, $B = (b_{ij})$ be two $M$ matrices of order $n$ such that $a_{ij} \leqslant b_{ij}$ for all $i, j$. Let $0 \leqslant t \leqslant 1$ and let $C = (c_{ij})$ be a real or complex square matrix of order $n$ such that

$$|c_{ii}| \geqslant ta_{ii} + (1 - t)b_{ii} \qquad \text{for all } i, \tag{4}$$

$$|c_{ij}| \leqslant |ta_{ij} + (1 - t)b_{ij}| \qquad \text{for } i \neq j. \tag{5}$$

From two earlier results [4, Proposition 3; 6, Theorem 4], it follows that the inequality

$$\left|\frac{C(\alpha \cap \beta)C(\alpha \cup \beta)}{C(\alpha)C(\beta)}\right| \geqslant \left[\frac{A(\alpha \cap \beta)A(\alpha \cup \beta)}{A(\alpha)A(\beta)}\right]^t \left[\frac{B(\alpha \cap \beta)B(\alpha \cup \beta)}{B(\alpha)B(\beta)}\right]^{1-t} \tag{6}$$

holds for any two subsets $\alpha, \beta$ of $\{1, 2, \ldots, n\}$. Thus the function $f$ defined by

$$f(\alpha) = t \log A(\alpha) + (1 - t) \log B(\alpha) - \log|C(\alpha)| \tag{7}$$

for $\alpha \subset \{1, 2, \ldots, n\}$ is a subadditive function on the lattice of subsets of the set $\{1, 2, \ldots, n\}$.

3. For general subadditive functions on an abstract distributive lattice, we gave in [7] an inequality which in the case of functions of type (3), yields a result stronger than Szász's inequality. In the following lines, we shall prove another inequality for subadditive functions on a distributive lattice. Inequality (10) below, i.e., the case $q = 1$ of (8), generalizes a determinantal inequality recently obtained by Carlson [3].

THEOREM. *If $f$ is a subadditive function defined on a distributive lattice $L$, then for any finite sequence $x_1, x_2, \ldots, x_p$ of elements of $L$, we have*

$$\sum_{1 \leqslant i_1 < \cdots < i_q \leqslant p} f(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_q}) \geqslant \sum_{k=q}^{p} \binom{k-1}{q-1} f(y_k) \qquad (1 \leqslant q \leqslant p), \tag{8}$$

*where*

$$y_k = \bigvee_{1 \leqslant i_1 < \cdots < i_k \leqslant p} (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}) \qquad (1 \leqslant k \leqslant p). \tag{9}$$

*Proof.* Consider first the case $q = 1$. In this case, (8) becomes

$$\sum_{k=1}^{p} f(x_k) \geqslant \sum_{k=1}^{p} f(y_k) \qquad (p \geqslant 1). \tag{10}$$

When $p = 1$, (10) is trivial. When $p = 2$, (10) is precisely the subadditive property. To prove (10) by induction on $p$, we may and shall assume $p \geqslant 3$. Let

$$z_k = \bigvee_{1 \leqslant i_1 < \cdots < i_k \leqslant p-1} (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}) \qquad (1 \leqslant k \leqslant p-1). \tag{11}$$

Then the inductive assumption gives us

$$\sum_{k=1}^{p-1} f(x_k) \geqslant \sum_{k=1}^{p-1} f(z_k). \tag{12}$$

Because $L$ is distributive,

$$z_k \vee (z_{k-1} \wedge x_p) = y_k \qquad (2 \leqslant k \leqslant p-1). \tag{13}$$

Also, since $z_k \leqslant z_{k-1}$,

$$z_k \wedge (z_{k-1} \wedge x_p) = z_k \wedge x_p \qquad (2 \leqslant k \leqslant p-1). \tag{14}$$

From (13), (14), and the subadditivity of $f$, we have

$$f(z_k) + f(z_{k-1} \wedge x_p) \geqslant f(y_k) + f(z_k \wedge x_p) \qquad (2 \leqslant k \leqslant p-1) \tag{15}$$

and therefore

$$\sum_{k=2}^{p-1} f(z_k) + f(z_1 \wedge x_p) \geqslant \sum_{k=2}^{p-1} f(y_k) + f(z_{p-1} \wedge x_p). \tag{16}$$

Combining (16) with

$$f(z_1) + f(x_p) \geqslant f(z_1 \vee x_p) + f(z_1 \wedge x_p) \tag{17}$$

and observing that $z_1 \vee x_p = y_1$, $z_{p-1} \wedge x_p = y_p$, we obtain

$$\sum_{k=1}^{p-1} f(z_k) + f(x_p) \geqslant \sum_{k=1}^{p} f(y_k). \tag{18}$$

Then (10) follows from (12) and (18). Inequality (8) is thus proved for $q = 1$ and all $p \geqslant 1$.

When $q = p$, both sides of (8) are equal to $f(x_1 \wedge x_2 \wedge \cdots \wedge x_p)$. To prove (8) for the case $(q, p)$ with $2 \leqslant q < p$, it suffices to show that the result for the cases $(q, p - 1)$ and $(q - 1, p - 1)$ implies the result for the case $(q, p)$.

Consider a fixed pair $(q, p)$ with $2 \leqslant q < p$. By the result for the case $(q, p - 1)$, we have

$$\sum_{1 \leqslant i_1 < \cdots < i_q \leqslant p-1} f(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_q}) \geqslant \sum_{k=q}^{p-1} \binom{k-1}{q-1} f(z_k), \tag{19}$$

where, as before, the $z_k$'s are defined by (11). If we apply the result for the case $(q - 1, p - 1)$ to the $p - 1$ elements $x_1 \wedge x_p, x_2 \wedge x_p, \ldots, x_{p-1} \wedge x_p$, then since $L$ is distributive, we get

$$\sum_{1 \leqslant i_1 < \cdots < i_{q-1} \leqslant p-1} f(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_{q-1}} \wedge x_p) \geqslant \sum_{k=q-1}^{p-1} \binom{k-1}{q-2} f(z_k \wedge x_p). \tag{20}$$

On the other hand, (15) gives us

$$\sum_{k=q}^{p-1} \binom{k-1}{q-1} [f(z_k) + f(z_{k-1} \wedge x_p)] \geqslant \sum_{k=q}^{p-1} \binom{k-1}{q-1} [f(y_k) + f(z_k \wedge x_p)]. \tag{21}$$

As $\binom{k}{q-1} = \binom{k-1}{q-1} + \binom{k-1}{q-2}$ for $k \geqslant q$ and $z_{p-1} \wedge x_p = y_p$, (21) may be written

$$\sum_{k=q}^{p-1} \binom{k-1}{q-1} f(z_k) + \sum_{k=q-1}^{p-1} \binom{k-1}{q-2} f(z_k \wedge x_p) \geqslant \sum_{k=q}^{p} \binom{k-1}{q-1} f(y_k). \tag{22}$$

Then the desired inequality (8) follows from (19), (20), and (22). This shows that for $2 \leqslant q < p$, the result for the case $(q, p)$ is implied by the result for the cases $(q, p - 1)$ and $(q - 1, p - 1)$. The theorem is thus proved for all $(q, p)$ with $1 \leqslant q \leqslant p$.

4. The hypothesis of the theorem is self-dual. That is, the hypothesis remains the same when the lattice operations $\vee$ and $\wedge$ are interchanged. Therefore, under the hypothesis of the above theorem, we have also

$$\sum_{1 \leqslant i_1 < \cdots < i_q \leqslant p} f(x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_q}) \geqslant \sum_{k=q}^{p} \binom{k-1}{q-1} f(u_k) \qquad (1 \leqslant q \leqslant p), \tag{23}$$

where

$$u_k = \bigwedge_{1 \leqslant i_1 < \cdots < i_k \leqslant p} (x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_k}) \qquad (1 \leqslant k \leqslant p). \tag{24}$$

If the lattice $L$ is not distributive, but the subadditive function $f$ is isotone, i.e., if $x \geqslant y$ in $L$ implies $f(x) \geqslant f(y)$, then inequality (10) remains valid. In fact, under the new hypothesis, (15) is still true, although (13) has to be replaced by $z_k \vee (z_{k-1} \wedge x_p) \geqslant y_k$.

For a function of the type (3), the theorem takes the following form. Let $A$ be a real or complex, square matrix of order $n$ such that its principal minors are all positive and satisfy inequality (2). Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be subsets of the set $\{1, 2, \ldots, n\}$. For $1 \leqslant k \leqslant p$, let $\beta_k$ be the set of those indices which are contained in at least $k$ of the sets $\alpha_1, \alpha_2, \ldots, \alpha_p$. Then

$$\prod_{1 \leqslant i_1 < \cdots < i_q \leqslant p} A(\alpha_{i_1} \cap \alpha_{i_2} \cap \cdots \cap \alpha_{i_q}) \geqslant \prod_{k=q}^{p} [A(\beta_k)]^{\binom{k-1}{q-1}} \qquad (1 \leqslant q \leqslant p). \tag{25}$$

In particular, if for some positive integer $r \leqslant p$, each of the indices $1, 2, \ldots, n$ is contained in exactly $r$ of the sets $\alpha_1, \alpha_2, \ldots, \alpha_p$, then

$$\prod_{1 \leqslant i_1 < \cdots < i_q \leqslant p} A(\alpha_{i_1} \cap \alpha_{i_2} \cap \cdots \cap \alpha_{i_q}) \geqslant (\det A)^{\binom{r}{q}} \qquad (1 \leqslant q \leqslant r). \tag{26}$$

In the case $q = 1$, (26) was first obtained by Marcus [10] for positive definite Hermitian matrices, then in [5] for $M$ matrices and totally positive matrices. This was recently generalized to the case $q = 1$ of (25) by Carlson [3].

As other examples of subadditive functions, we mention the dimension function on a semimodular lattice [1, p. 100], and the upper integral on a locally compact space with respect to a positive measure [2, p. 109].

## REFERENCES

1 G. Birkhoff, Lattice Theory, *Amer. Math. Soc. Colloq. Publ. XXV*, New York, 1948.
2 N. Bourbaki, *Intégration*, Chap. I–IV, Hermann, Paris, 1952.
3 D. Carlson, Note on some determinantal inequalities, to appear.
4 K. Fan, Note on $M$-matrices, *Quart. J. Math. Oxford* (2), **11**(1960), 43–49.
5 K. Fan, Some matrix inequalities, *Abh. Math. Sem. Univ. Hamburg* **29**(1966), 185–196.

6 K. Fan, Inequalities for the sum of two $M$-matrices, *Inequalities: Proceedings of a Symposium* (O. Shisha, ed.), Academic Press, New York, 1967.
7 K. Fan, Subadditive functions on a distributive lattice and an extension of Szász's inequality, *J. Math. Analysis Appl.* **18**(1967), 262–268.
8 F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin, 1960.
9 D. M. Koteljanskiĭ, The theory of nonnegative and oscillating matrices (Russian), *Ukrain. Mat. Ž.* 2(1950), 94–101; English transl.: *Amer. Math. Soc. Transl.* (2), **27**(1963), 1–8.
10 M. Marcus, Matrix applications of a quadratic identity for decomposable symmetrized tensors, *Bull. Amer. Math. Soc.* **71**(1965), 360–364.
11 A. M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale, *Comment. Math. Helvetici* **10**(1937), 69–96.
12 A. M. Ostrowski, Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen, *Comment. Math. Helvetici* **30**(1956), 175–210.

*Received May 19, 1967*